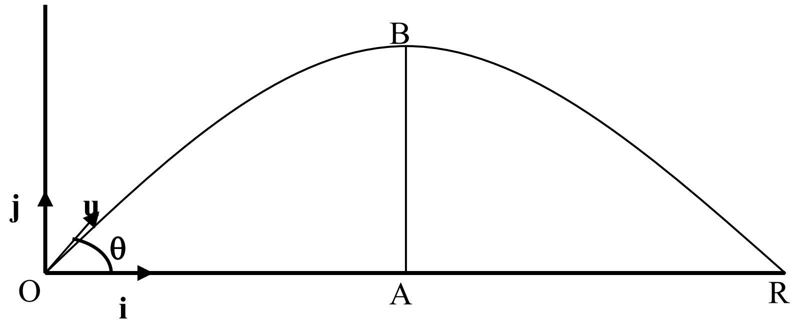
Mathematics of Simple Trajectory of Projectile



Conditions

1. A mass is projected from **O** with initial velocity **u** that makes with an angle θ with the vector **i** in the x-axis direction. It reaches a maximum height at point B vertically above point A and touches the ground at the point R.
2. Suppose there is no air resistance to the mass and the vertical acceleration is $-g$ in the direction of y-axis.

Vectors

Acceleration of mass, $\boldsymbol{a} = 0\boldsymbol{i} - g\boldsymbol{j}$ **(1)**

Initial Velocity vector, $\boldsymbol{v} = u\cos\theta\,\boldsymbol{i} + u\sin\theta\,\boldsymbol{j}$ **(2)**

Displacement of mass at time t, $\boldsymbol{s} = ut\cos\theta\,\boldsymbol{i} + [ut\sin\theta - \tfrac{1}{2}gt^2]\,\boldsymbol{j}$ **(3)**

By differentiating (3), or otherwise, we get:

Velocity of mass at time t, $\boldsymbol{v} = u\cos\theta\,\boldsymbol{i} + [u\sin\theta - gt]\,\boldsymbol{j}$ **(4)**

Time for the mass to touch the ground again

In (3), put $\mathbf{s_y} = 0$, $\therefore ut\sin\theta - \tfrac{1}{2}gt^2 = 0$.

Since $t \neq 0$, Final time t_f is given by: $\therefore t_f = \dfrac{2u\sin\theta}{g}$ **(5)**

Maximum vertical height

From (5), the time for the mass to reach the maximum height $= t_{\max} = \dfrac{t_f}{2} = \dfrac{u\sin\theta}{g}$ **(6)**

Max. height = AB = $\mathbf{s_y}$ **($\mathbf{t = t_{max}}$)** $= u\left(\dfrac{u\sin\theta}{g}\right)\sin\theta - \dfrac{1}{2}g\left(\dfrac{u\sin\theta}{g}\right)^2 = \dfrac{u^2\sin^2\theta}{2g}$ **(7)**

Horizontal distance traveled

Horizontal range = OR = $\mathbf{s_x}$ **(t = t$_f$)** $= u\dfrac{2u\sin\theta}{g}\cos\theta = \dfrac{u^2\sin 2\theta}{g}$ **(8)**

Half horizontal range = OA = $\dfrac{u^2\sin 2\theta}{2g}$ **(9)**

By using different angle of projectile θ, we can change the horizontal distance OR.

Since the maximum of sin 2θ = 1, when θ = 45°, ∴ Max. of OR $= \dfrac{u^2}{g}$ **(10)**

Equation of trajectory

From (3), $x = ut\cos\theta$ **(11)**

$y = ut\sin\theta - ½gt^2$ **(12)**

From (11), $t = \dfrac{x}{u\cos\theta}$ **(13)**

(13)↓(12), $y = \dfrac{\sin\theta}{\cos\theta}x - \dfrac{g}{2u^2\cos^2\theta}x^2$ or $y = (\tan\theta)x - \left(\dfrac{g\sec^2\theta}{2u^2}\right)x^2$ **(14)**

(14) is an equation of a **parabola**.

The vertex of (14) is $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right) = \left(\dfrac{u^2\sin 2\theta}{2g}, \dfrac{u^2\sin^2\theta}{2g}\right)$, which is in line with (8) and (7).

Arc Length of trajectory

The arc length, $L = \int_0^{t_f}\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt = \int_0^{t_f}\sqrt{(v_x)^2 + (v_y)^2}\,dt = \int_0^{t_f}\sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}\,dt$

Putting $a = u\cos\theta$, $z = gt - u\sin\theta$, $dz = gdt$

When $t = t_f$, by (5), $z = u\sin\theta$.

When $t = 0$, $z = -u\sin\theta$.

$\therefore L = \dfrac{1}{2g}\int_{-u\sin\theta}^{u\sin\theta}\sqrt{a^2 + z^2}\,dz$, which is a standard integral (see below).

$$= \frac{1}{2g}\left[z\sqrt{a^2+z^2} + a^2\ln\left|z + \sqrt{a^2+z^2}\right|\right]_{-u\sin\theta}^{u\sin\theta}$$

$$= \frac{1}{2g}\left[(u\sin\theta)\sqrt{(u\cos\theta)^2 + (u\sin\theta)^2} + (u\cos\theta)^2\ln\left|(u\sin\theta) + \sqrt{(u\cos\theta)^2 + (u\sin\theta)^2}\right|\right]$$

$$-\frac{1}{g}\left[-(u\sin\theta)\sqrt{(u\cos\theta)^2 + (-u\sin\theta)^2} + (u\cos\theta)^2\ln\left|-(u\sin\theta) + \sqrt{(u\cos\theta)^2 + (-u\sin\theta)^2}\right|\right]$$

$$= \frac{u^2\sin\theta}{2g} + \frac{u^2\cos\theta}{2g}\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| \qquad \textbf{(15)}$$

Evaluate the integral:

Let $\mathrm{I} = \int \sqrt{\mathrm{a}^2 + \mathrm{z}^2}\mathrm{dz} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} - \int \mathrm{z}\mathrm{d}\left(\sqrt{\mathrm{a}^2 + \mathrm{z}^2}\right)$ (integration by parts)

$$= \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} - \int \mathrm{z}\frac{2\mathrm{z}}{2\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\mathrm{dz} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} - \int \frac{(\mathrm{a}^2+\mathrm{z}^2)-\mathrm{a}^2}{\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\mathrm{dz}$$

$$= \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} - \int \sqrt{\mathrm{a}^2 + \mathrm{z}^2}\mathrm{dz} + \int \frac{\mathrm{a}^2}{\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\mathrm{dz}$$

$$\mathrm{I} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} - \mathrm{I} + \mathrm{a}^2\int \frac{1}{\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\mathrm{dz}$$

$$2\mathrm{I} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} + \mathrm{a}^2\int \frac{1}{\mathrm{z}+\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\frac{\mathrm{z}+\sqrt{\mathrm{a}^2+\mathrm{z}^2}}{\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\mathrm{dz}$$

$$2\mathrm{I} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} + \mathrm{a}^2\int \frac{1}{\mathrm{z}+\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\left(1 + \frac{\mathrm{z}}{\sqrt{\mathrm{a}^2+\mathrm{z}^2}}\right)\mathrm{dz}$$

$$2\mathrm{I} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} + \mathrm{a}^2\int \frac{\mathrm{d}\left(\mathrm{z}+\sqrt{\mathrm{a}^2+\mathrm{z}^2}\right)}{\mathrm{z}+\sqrt{\mathrm{a}^2+\mathrm{z}^2}}$$

$$2\mathrm{I} = \mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} + \mathrm{a}^2\ln\left|\mathrm{z} + \sqrt{\mathrm{a}^2 + \mathrm{z}^2}\right|$$

$$\mathrm{I} = \frac{1}{2}\left[\mathrm{z}\sqrt{\mathrm{a}^2 + \mathrm{z}^2} + \mathrm{a}^2\ln\left|\mathrm{z} + \sqrt{\mathrm{a}^2 + \mathrm{z}^2}\right|\right] + \mathrm{C}$$

<u>Area enclosed by the trajectory and x-axis</u>

The area $A = \int_0^{t_f} y\frac{dx}{dt}dt = \int_0^{t_f}\left(ut\sin\theta - \frac{1}{2}gt^2\right)(u\cos\theta)\,dt$

$$= \int_0^{t_f}(u^2t\sin\theta\cos\theta)\,dt - \frac{1}{2}\int_0^{t_f}(gu\ \ t^2\cos\theta)dt$$

$$= \left[(u^2\sin\theta\cos\theta)\frac{t^2}{2} - \frac{1}{2}(gu\cos\theta)\frac{t^3}{3}\right]_0^{t_f}$$

By (5), $t_f = \frac{2u\sin\theta}{g}$

$$A = \left[(u^2\sin\theta\cos\theta)\frac{t^2}{2} - \frac{1}{2}(gu\cos\theta)\frac{t^3}{3}\right]_0^{\frac{2u\sin\theta}{g}}$$

$$A = \frac{1}{2}(u^2\sin\theta\cos\theta)\left(\frac{2u\sin\theta}{g}\right)^2 - \frac{1}{6}(gu\cos\theta)\left(\frac{2u\sin\theta}{g}\right)^3 = \frac{2u^4}{5g^2}\cos\theta\sin^3\theta$$